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On Some Properties of $\mathcal{I}_2$-Convergence and $\mathcal{I}_2$-Cauchy of Double Sequences

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Abstract

In this paper we study the concepts of $\mathcal{I}$-Cauchy and $\mathcal{I}^*$-Cauchy for double sequences in a linear metric space. Also, we give the relation between $\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence of double sequences of functions defined between linear metric spaces.

Keywords: *Double sequence, ideal, $\mathcal{I}$-Cauchy, $\mathcal{I}$-convergence.*

1 Introduction

The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [7] and Schoenberg [25]. A lot of developments have been made in this area after the works of Śalát [23] and Fridy [10, 11]. In general, statistically convergent sequences satisfy many of the properties of ordinary convergent sequences in metric spaces [7, 10, 11, 22]. This concept was extended to the double sequences by Mursaleen and Edely [18]. Çakan and Altay [3] presented multidimensional analogues of the results of Fridy and Orhan [12]. They introduced the concepts of statistically boundedness, statistical inferior and statistical superior of double sequences. In addition to these results they investigated statistical core for double sequences and studied an inequality related to the statistical core and P-core of bounded double sequences. Furthermore, Gökhan et al. [14] introduced the notion of pointwise and uniform statistical convergent of double sequences of real-valued functions.

The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [15] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subsets of the set of natural numbers. Nuray and Ruckle [20] independently introduced the same concept as the name generalized statistical convergence. Kostyrko et al. [16] gave some of basic properties of $\mathcal{I}$-convergence and dealt with extremal $\mathcal{I}$-limit points. Das et al. [4] introduced the concept of $\mathcal{I}$-convergence of double sequences in a metric space and studied some properties of this convergence. Also, Das and Malik [5] defined the concept of $\mathcal{I}$-limit points, $\mathcal{I}$-cluster points, $\mathcal{I}$-limit superior and $\mathcal{I}$-limit inferior of double sequences.

Nabiev et al. [19] proved a decomposition theorem for $\mathcal{I}$-convergent sequences and introduced the notions of $\mathcal{I}$-Cauchy sequence and $\mathcal{I}^*$-Cauchy sequence, and then studied their certain properties. Balcerzak et al. [2] discussed various kinds of statistical convergence and $\mathcal{I}$-convergence for sequences of real valued functions or for sequences of functions into a metric space. For real valued measurable functions defined on a measure space $(X, \mathfrak{M}, \mu)$, they obtained a statistical version of the Egorov theorem (when $\mu(X) < \infty$). They showed that, in its assertion, equi-statistical convergence on a big set cannot be replaced by uniform statistical convergence. Also, they considered statistical convergence in measure and $\mathcal{I}$-convergence in measure, with some consequences of the Riesz theorem. Gezer and Karakuş [13] investigated $\mathcal{I}$-pointwise and uniform convergence and $\mathcal{I}^*$-pointwise and uniform convergence of function sequences and then they examined the relation between them. A lot of developments have been made in this area after the works of [2, 6, 8, 17, 24, 26].

In this paper first we investigate some properties of $\mathcal{I}$-Cauchy, $\mathcal{I}^*$-Cauchy of double sequences in a linear metric space with property $(AP2)$. Next we prove two theorems of $\mathcal{I}$-convergent, $\mathcal{I}^*$-convergent of double sequences of functions on metric space with property $(AP2)$.

2 Definitions and Notations

Throughout the paper by $\mathbb{N}$ and $\mathbb{R}$, we denote the sets of all positive integers and real numbers, and χ_A is the the characteristic function of $A \subset \mathbb{N}$.

Now, we recall some concepts of the sequences (See [4, 7, 14, 15, 18, 21, 24]).

A subset A of $\mathbb{N}$ is said to have asymptotic density $d(A)$ if

$$d(A) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_A(k).$$

A sequence $x = (x_n)_{n\in\mathbb{N}}$ of real numbers is said to be statistically convergent to $L \in \mathbb{R}$ if for any $\varepsilon > 0$ we have $d(A(\varepsilon)) = 0$, where $A(\varepsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}$.

A double sequence $x = (x_{mn})_{m,n\in\mathbb{N}}$ of real numbers is said to be convergent to $L \in \mathbb{R}$ in Pringsheim's sense if for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_{mn} - L| < \varepsilon$, whenever $m, n > N_\varepsilon$. In this case we write

$$\lim_{m,n\to\infty} x_{mn} = L.$$

A double sequence $x = (x_{mn})_{m,n\in\mathbb{N}}$ of real numbers is said to be bounded if there exists a positive real number M such that $|x_{mn}| < M$, for all $m, n \in \mathbb{N}$.

Let $K \subset \mathbb{N} \times \mathbb{N}$. Let K_{mn} be the number of $(j,k) \in K$ such that $j \leq m$, $k \leq n$. If the sequence $\left\{\frac{K_{mn}}{m.n}\right\}$ has a limit in Pringsheim's sense then we say that K has double natural density and is denoted by

$$d_2(K_{mn}) = \lim_{m,n\to\infty} \frac{K_{mn}}{m.n}.$$

A double sequence $x = (x_{mn})_{m,n\in\mathbb{N}}$ of real numbers is said to be statistically convergent to $L \in \mathbb{R}$, if for any $\varepsilon > 0$ we have $d_2(A(\varepsilon)) = 0$, where $A(\varepsilon) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : |x_{mn} - L| \geq \varepsilon\}$.

A double sequence of functions $\{f_{mn}\}$ is said to be pointwise convergent to f on a set $S \subset \mathbb{R}$, if for each point $x \in S$ and for each $\varepsilon > 0$, there exists a positive integer $N(x, \varepsilon)$ such that

$$|f_{mn}(x) - f(x)| < \varepsilon,$$

for all $m, n > N$. We write

$$\lim_{m,n\to\infty} f_{mn}(x) = f(x) \ \textit{or} \ f_{mn} \to f,$$

on S.

A double sequence of functions $\{f_{mn}\}$ is said to be pointwise statistically convergent to f on a set $S \subset \mathbb{R}$, if for every $\varepsilon > 0$,

$$\lim_{m,n\to\infty} \frac{1}{mn} |\{(i,j), i \leq m \ \textit{and} \ j \leq n : |f_{ij}(x) - f(x)| \geq \varepsilon\}| = 0,$$

for each (fixed) $x \in S$, i.e., for each (fixed) $x \in S$,

$$|f_{ij}(x) - f(x)| < \varepsilon, \ \ a.a.\,(i,j).$$

We write

$$st - \lim_{m,n\to\infty} f_{mn}(x) = f(x) \ \textit{or} \ f_{mn} \to_{st} f,$$

on S.

Let $X \neq \emptyset$. A class $\mathcal{I}$ of subsets of X is said to be an ideal in X provided the following statements hold:

(i) $\emptyset \in \mathcal{I}$.

(ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$.

(iii) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$.

$\mathcal{I}$ is called a nontrivial ideal if $X \notin \mathcal{I}$.

Let $X \neq \emptyset$. A non empty class $\mathcal{F}$ of subsets of X is said to be a filter in X provided the following statements hold:

(i) $\emptyset \notin \mathcal{F}$.

(ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$.

(iii) $A \in \mathcal{F}$, $A \subset B$ implies $B \in \mathcal{F}$.

If $\mathcal{I}$ is a nontrivial ideal in X, $X \neq \emptyset$, then the class

$$\mathcal{F}(\mathcal{I}) = \{M \subset X : (\exists A \in \mathcal{I})(M = X \backslash A)\}$$

is a filter on X, called the filter associated with $\mathcal{I}$.

A nontrivial ideal $\mathcal{I}$ in X is called admissible if $\{x\} \in \mathcal{I}$ for each $x \in X$.

Throughout the paper we take $\mathcal{I}_2$ as a nontrivial admissible ideal in $\mathbb{N} \times \mathbb{N}$.

A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$ for each $i \in N$.

It is evident that a strongly admissible ideal is also admissible.

Let $\mathcal{I}_2^0 = \{A \subset \mathbb{N} \times \mathbb{N} : (\exists m(A) \in \mathbb{N})(i, j \geq m(A) \Rightarrow (i, j) \notin A)\}$. Then $\mathcal{I}_2^0$ is a nontrivial strongly admissible ideal and clearly an ideal $\mathcal{I}_2$ is strongly admissible if and only if $\mathcal{I}_2^0 \subset \mathcal{I}_2$.

In this study we consider the $\mathcal{I}_2$ and $\mathcal{I}_2^*$-convergence of double sequences in the more general structure of a metric space (X, ρ). Unless stated otherwise we shall denote the metric space (X, ρ) in short by X.

Let (X, ρ) be a linear metric space and $\mathcal{I}_2 \subset 2^{\mathbb{N} \times \mathbb{N}}$ be a strongly admissible ideal. A double sequence $x = (x_{mn})_{m,n \in \mathbb{N}}$ of elements of X is said to be $\mathcal{I}_2$-convergent to $L \in X$, if for any $\varepsilon > 0$ we have

$$A(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, L) \geq \varepsilon\} \in \mathcal{I}_2.$$

In this case we say that x is $\mathcal{I}_2$-convergent and we write

$$\mathcal{I}_2 - \lim_{m,n \to \infty} x_{mn} = L.$$

If $\mathcal{I}_2$ is a strongly admissible ideal on $\mathbb{N} \times \mathbb{N}$, then usual convergence implies $\mathcal{I}_2$-convergence.

Let (X, ρ) be a linear metric space and $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. A double sequence $x = (x_{mn})_{m,n\in\mathbb{N}}$ of elements of X is said to be $\mathcal{I}_2^*$-convergent to $L \in X$, if there exists a set $M \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $\mathbb{N} \times \mathbb{N} \backslash M \in \mathcal{I}_2$) such that

$$\lim_{m,n\to\infty} x_{mn} = L,$$

for $(m, n) \in M$ and we write

$$\mathcal{I}_2^* - \lim_{m,n\to\infty} x_{mn} = L.$$

Let (X, ρ) be a linear metric space and $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. A double sequence $x = (x_{mn})_{m,n\in\mathbb{N}}$ of elements of X is said to be $\mathcal{I}_2$-Cauchy if for every $\varepsilon > 0$ there exist $s = s(\varepsilon), t = t(\varepsilon) \in \mathbb{N}$ such that

$$A(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, x_{st}) \geq \varepsilon\} \in \mathcal{I}_2.$$

We say that an admissible ideal $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ satisfies the property (AP2), if for every countable family of mutually disjoint sets $\{A_1, A_2, ...\}$ belonging to $\mathcal{I}_2$, there exists a countable family of sets $\{B_1, B_2, ...\}$ such that $A_j \Delta B_j \in \mathcal{I}_2^0$, i.e., $A_j \Delta B_j$ is included in the finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$ for each $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}_2$ (hence $B_j \in \mathcal{I}_2$ for each $j \in \mathbb{N}$).

Now we begin with quoting the lemmas due to Das et al. [4] which are needed in the proof of theorems.

Lemma 2.1 *[4, Theorem 1], Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. If $\mathcal{I}^*{}_2 - \lim_{m,n\to\infty} x_{mn} = L$ then $\mathcal{I}_2 - \lim_{m,n\to\infty} x_{mn} = L$.*

Lemma 2.2 *[4, Theorem 3], If $\mathcal{I}_2$ is an admissible ideal of $\mathbb{N}\times\mathbb{N}$ having the property (AP2) and (X, ρ) is an arbitrary metric space, then for an arbitrary double sequence $x = (x_{mn})_{m,n\in\mathbb{N}}$ in X*

$$\mathcal{I}_2 - \lim_{m,n\to\infty} x_{mn} = L \textit{ implies } \mathcal{I}_2^* - \lim_{m,n\to\infty} x_{mn} = L$$

3 $\mathcal{I}_2$ and $\mathcal{I}_2^*$-Cauchy Of Double Sequences

Definition 3.1 ([9]) *Let (X, ρ) be a linear metric space and $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. A double sequence $x = (x_{mn})$ in X is said to be $\mathcal{I}_2^*$-Cauchy sequence if there exists a set $M \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $H = \mathbb{N}\times\mathbb{N}\backslash M \in \mathcal{I}_2$) such that for every $\varepsilon > 0$ and for $(m, n), (s, t) \in M$, $m, n, s, t > k_0 = k_0(\varepsilon)$*

$$\rho(x_{mn}, x_{st}) < \varepsilon.$$

In this case we write

$$\lim_{m,n,s,t\to\infty} \rho(x_{mn}, x_{st}) = 0.$$

Lemma 3.2 ([9]) *Let (X, ρ) be a linear metric space and $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a stronly admissible ideal. If $x = (x_{mn})$ in X is an $\mathcal{I}_2^*$-Cauchy sequence then it is $\mathcal{I}_2$-Cauchy.*

Theorem 3.3 *Let $\{P_i\}_{i=1}^{\infty}$ be a countable collection of subsets of $\mathbb{N} \times \mathbb{N}$ such that $\{P_i\}_{i=1}^{\infty} \in F(\mathcal{I}_2)$ for each i, where $\mathcal{F}(\mathcal{I}_2)$ is a filter associate with a strongly admissible ideal $\mathcal{I}_2$ with the property (AP2). Then there exists a set $P \subset \mathbb{N} \times \mathbb{N}$ such that $P \in \mathcal{F}(\mathcal{I}_2)$ and the set $P \backslash P_i$ is finite for all i.*

Proof. Let $A_1 = \mathbb{N} \times \mathbb{N} \backslash P_1$, $A_m = (\mathbb{N} \times \mathbb{N} \backslash P_m) \backslash (A_1 \cup A_2 \cup ... \cup A_{m-1})$, $(m = 2, 3, ...)$. It is easy to see that $A_i \in \mathcal{I}_2$ for each i and $A_i \cap A_j = \emptyset$, when $i \neq j$. Then by (AP2) property of $\mathcal{I}_2$ we conclude that there exists a countable family of sets $\{B_1, B_2, ...\}$ such that $A_j \triangle B_j \in \mathcal{I}_2^0$, i.e., $A_j \triangle B_j$ is included in finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$ for each j and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}_2$. Put $P = \mathbb{N} \times \mathbb{N} \backslash B$. It is clear that $P \in \mathcal{F}(\mathcal{I}_2)$.

Now we prove that the set $P \backslash P_i$ is finite for each i. Assume that there exists a $j_0 \in \mathbb{N}$ such that $P \backslash P_{j_0}$ has infinitely many elements. Since each $A_j \triangle B_j$ $(j = 1, 2, 3, ..., j_0)$ are included in finite union of rows and columns, there exists $m_0, n_0 \in \mathbb{N}$ such that

$$(\bigcup_{j=1}^{j_0} B_j) \cap C_{m_0 n_0} = (\bigcup_{j=1}^{j_0} A_j) \cap C_{m_0 n_0}, \tag{1}$$

where $C_{m_0 n_0} = \{(m, n) : m \geq m_0 \wedge n \geq n_0\}$. If $m \geq m_0 \wedge n \geq n_0$ and $(m, n) \notin B$, then

$$(m, n) \notin \bigcup_{j=1}^{j_0} B_j$$

and so by (1)

$$(m, n) \notin \bigcup_{j=1}^{j_0} A_j.$$

Since $A_{j_0} = (\mathbb{N} \times \mathbb{N} \backslash P_{j_0}) \backslash \bigcup_{j=1}^{j_0 - 1} A_j$ and $(m, n) \notin A_{j_0}$, $(m, n) \notin \bigcup_{j=1}^{j_0 - 1} A_j$ we have $(m, n) \in P_{j_0}$ for $m \geq m_0 \wedge n \geq n_0$. Therefore, for all $m \geq m_0 \wedge n \geq n_0$ we get $(m, n) \in P$ and $(m, n) \in P_{j_0}$. This shows that the set $P \backslash P_{j_0}$ has a finite number of elements. This contradicts our assumption that the set $P \backslash P_{j_0}$ is an infinite set.

Theorem 3.4 *Let (X, ρ) be a linear metric space. If $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ is a strongly admissible ideal with the property (AP2) then the concepts $\mathcal{I}_2$-Cauchy double sequence and $\mathcal{I}_2^*$-Cauchy double sequence coincide.*

Proof. If a double sequence is $\mathcal{I}_2^*$-Cauchy, then it is $\mathcal{I}_2$-Cauchy by Lemma 3.2, where $\mathcal{I}_2$ need not have the property $(AP2)$.

Now it is sufficient to prove that a double sequence $x = (x_{mn})$ in X is a $\mathcal{I}_2^*$-Cauchy double sequence under assumption that it is an $\mathcal{I}_2$-Cauchy double sequence. Let $x = (x_{mn})$ in X be an $\mathcal{I}_2$-Cauchy double sequence. Then, there exists $s = s(\varepsilon), t = t(\varepsilon) \in \mathbb{N}$ such that

$$A(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, x_{st}) \geq \varepsilon\} \in \mathcal{I}_2, \text{ for every } \varepsilon > 0.$$

Let

$$P_i = \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, x_{s_it_i}) < \frac{1}{i}\right\}; \quad (i = 1, 2, \ldots),$$

where $s_i = s(1\backslash i), t_i = t(1\backslash i)$. It is clear that $P_i \in \mathcal{F}(\mathcal{I}_2)$, $(i = 1, 2, \ldots)$. Since $\mathcal{I}_2$ has the property $(AP2)$, then by Theorem 3.3 there exists a set $P \subset \mathbb{N} \times \mathbb{N}$ such that $P \in \mathcal{F}(\mathcal{I}_2)$, and $P \backslash P_i$ is finite for all i. Now we show that

$$\lim_{m,n,s,t \to \infty} \rho(x_{mn}, x_{st}) = 0,$$

for $(m,n), (s,t) \in P$. To prove this, let $\varepsilon > 0$ and $j \in \mathbb{N}$ such that $j > 2/\varepsilon$. If $(m,n), (s,t) \in P$ then $P \backslash P_i$ is a finite set, so there exists $k = k(j)$ such that $(m,n), (s,t) \in P_j$ for all $m, n, s, t > k(j)$. Therefore,

$$\rho(x_{mn}, x_{s_it_i}) < \frac{1}{j} \ \textit{and} \ \rho(x_{st}, x_{s_it_i}) < \frac{1}{j},$$

for all $m, n, s, t > k(j)$. Hence it follows that

$$\rho(x_{mn}, x_{st}) \leq \rho(x_{mn}, x_{s_it_i}) + \rho(x_{st}, x_{s_it_i}) < \frac{1}{j} + \frac{1}{j} = \frac{2}{j} < \varepsilon,$$

for all $n, m, s, t > k(j)$. Thus, for any $\varepsilon > 0$ there exists $k = k(\varepsilon)$ such that for $m, n, s, t > k(\varepsilon)$ and $(m,n), (s,t) \in P$

$$\rho(x_{mn}, x_{st}) < \varepsilon.$$

This shows that the sequence (x_{mn}) is an $\mathcal{I}_2^*$-Cauchy sequence.

4 $\mathcal{I}_2$ And $\mathcal{I}_2^*$-Convergence For Double Sequences Of Functions.

Definition 4.1 *Let $\mathcal{I}_2 \subset 2^{\mathbb{N} \times \mathbb{N}}$ be a strongly admissible ideal, (X, d_x) and (Y, d_y) two linear metric spaces, $f_{mn} : X \to Y$ a double sequence of functions* *and $f : X \to Y$. A double sequence of functions $\{f_{mn}\}$ is said to be pointwise $\mathcal{I}_2$-convergent to f on X, if for every $\varepsilon > 0$*

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : d_y(f_{mn}(x) - f(x)) \geq \varepsilon\} \in \mathcal{I}_2,$$

for each $x \in X$. This can be written by the formula

$$(\forall x \in X) \ \ (\forall \varepsilon > 0) \ \ (\exists H \in \mathcal{I}_2) \ \ (\forall (m,n) \notin H) \ \ d_y(f_{mn}(x) - f(x)) < \varepsilon.$$

This convergence can be showed by

$$f_{mn} \to_{\mathcal{I}_2} f \ (pointwise).$$

The function f is called the double $\mathcal{I}_2$-limit function of the sequence $\{f_{mn}\}$.

A double sequence of functions $\{f_{mn}\}$ is said to be pointwise $\mathcal{I}_2^*$- convergent to f on X if there exists a set $M \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $H = \mathbb{N} \times \mathbb{N} \backslash M \in \mathcal{I}_2$) such that

$$\lim_{m,n\to\infty} f_{mn}(x) = f(x),$$

for $(m,n) \in M$ and for each $x \in X$ and we write

$$\mathcal{I}_2^* - \lim_{m,n\to\infty} f_{mn} = f \text{ or } f_{mn} \to_{\mathcal{I}_2^*} f.$$

Theorem 4.2 *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal having the property (AP2), (X, d_x) and (Y, d_y) two linear metric spaces, $f_{mn} : X \to Y$ a double sequence of functions and $f : X \to Y$. If $\{f_{mn}\}$ double sequence of functions is $\mathcal{I}_2$-convergent then it is $\mathcal{I}_2^*$-convergent.*

Proof. Let $\mathcal{I}_2$ satisfy the property $(AP2)$ and $\mathcal{I}_2 - \lim_{m,n\to\infty} f_{mn}(x) = f(x)$ for each $x \in X$. Then for any $\varepsilon > 0$

$$A(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : d_y(f_{mn}(x), f(x)) \geq \varepsilon\} \in \mathcal{I}_2,$$

for each $x \in X$. Now put

$$A_1 = \{(m,n) \in \mathbb{N} \times \mathbb{N} : d_y(f_{mn}(x), f(x)) \geq 1\}$$

and

$$A_k = \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{k} \leq d_y(f_{mn}(x), f(x)) < \frac{1}{k-1}\right\},$$

for $k \geq 2$. It is clear that $A_i \cap A_j = \emptyset$ for $i \neq j$ and $A_i \in \mathcal{I}_2$ for each $i \in \mathbb{N}$. By virtue of $(AP2)$ there exists a sequence $\{B_k\}_{k\in\mathbb{N}}$ of sets such that $A_j \triangle B_j$

is included in finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$ for each $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}_2$.

We shall prove that

$$\lim_{\substack{m,n\to\infty \\ (m,n)\in M}} f_{mn}(x) = f(x),$$

for $M = \mathbb{N} \times \mathbb{N} \backslash B \in \mathcal{F}(\mathcal{I}_2)$. Let $\delta > 0$ be given. Choose $k \in \mathbb{N}$ such that $\frac{1}{k} < \delta$. Then, we have

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : d_y(f_{mn}(x), f(x)) \geq \delta\} \subset \bigcup_{j=1}^{k} A_j.$$

Since $A_j \triangle B_j$, $j = 1, 2, ..., k$, are included in finite union of rows and columns, there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} \left(\bigcup_{j=1}^{k} B_j\right) & \cap \quad \{(m,n) : m \geq n_0 \wedge n \geq n_0\} \\ & = \quad \left(\bigcup_{j=1}^{k} A_j\right) \cap \{(m,n) : m \geq n_0 \wedge n \geq n_0\}. \end{aligned}$$

If $m, n \geq n_0$ and $(m,n) \notin B$ then

$$(m,n) \notin \bigcup_{j=1}^{k} B_j \text{ and so } (m,n) \notin \bigcup_{j=1}^{k} A_j.$$

Thus we have $d_y(f_{mn}(x), f(x)) < \frac{1}{k} < \delta$, for each $x \in X$. This implies that

$$\lim_{\substack{m,n\to\infty \\ (m,n)\in M}} f_{mn}(x) = f(x),$$

for each $x \in X$.

For the converse we have the following theorem.

Theorem 4.3 *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal, (X, d_x) and (Y, d_y) two linear metric spaces, $f_{mn} : X \to Y$ a double sequence of functions and $f : X \to Y$. If Y has at least one accumulation point and*

$$\mathcal{I}_2 - \lim_{m,n\to\infty} f_{mn}(x) = f(x) \textit{ implies } \mathcal{I}_2^* - \lim_{m,n\to\infty} f_{mn}(x) = f(x),$$

for each $x \in X$, then $\mathcal{I}_2$ has the property (AP2) on $\mathbb{N} \times \mathbb{N}$.

Proof. Suppose that $f(x)$ is an accumulation point of Y for $x \in X$. There exists a sequence $(h_k)_{k\in\mathbb{N}}$ of distinct functions on X such that $h_k \neq f$ for any $k \in \mathbb{N}$, $f(x) = \lim_{k\to\infty} h_k(x)$ and the sequence $\{d_y(h_k(x), f(x))\}_{k\in\mathbb{N}}$ is a decreasing sequence converging to zero for each $x \in X$. We define

$$g_k(x) = d_y(h_k(x), f(x)),$$

for $k \in \mathbb{N}$ and for each $x \in X$. Let $\{A_j\}_{j\in\mathbb{N}}$ be a disjoint family of nonempty sets from $\mathcal{I}_2$. Define a sequence $\{f_{mn}\}$ in the following way:
(i) $f_{mn}(x) = h_j(x)$ if $(m,n) \in A_j$ and
(ii) $f_{mn}(x) = f(x)$ if $(m,n) \notin A_j$,
for any j and for each $x \in X$.

Let $\delta > 0$ be given. Choose $k \in \mathbb{N}$ such that $g_k(x) < \delta$, for each $x \in X$. Then we have

$$A(\delta) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : d_y(f_{mn}(x), f(x)) \geq \delta\} \subset A_1 \cup A_2 \cup \cdots \cup A_k$$

and so $A(\delta) \in \mathcal{I}_2$. This implies

$$\mathcal{I}_2 - \lim_{m,n\to\infty} f_{mn}(x) = f(x),$$

for each $x \in X$. By virtue of our assumption, we have

$$\mathcal{I}_2^* - \lim_{m,n\to\infty} f_{mn}(x) = f(x),$$

for each $x \in X$. Hence there exists a set $M = \mathbb{N}\times\mathbb{N}\backslash B \in \mathcal{F}(\mathcal{I}_2)$, $(B \in \mathcal{I}_2)$ such that

$$\lim_{\substack{m,n\to\infty \\ (m,n)\in M}} f_{mn}(x) = f(x), \tag{2}$$

for each $x \in X$. Let $B_j = A_j \cap B$ for each $j \in \mathbb{N}$. So we have $B_j \in \mathcal{I}_2$ and

$$\bigcup_{j=1}^{\infty} B_j = B \cap \bigcup_{j=1}^{\infty} A_j \subset B \text{ and so } \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}_2,$$

for each $j \in \mathbb{N}$. If $A_j \cap M$ is not included in the finite union of rows and columns in $\mathbb{N}\times\mathbb{N}$ for fix $j \in \mathbb{N}$, then M must contain an infinite sequence of elements $\{(m_k, n_k)\}$ where both $m_k, n_k \to \infty$ and

$$f_{m_k n_k}(x) = h_j(x) \neq f(x),$$

for all $k \in \mathbb{N}$ and for each $x \in X$, which contradicts (2). Hence $A_j \cap M$ must be contained in the finite union of rows and columns in $\mathbb{N}\times\mathbb{N}$. Thus

$$A_j \bigtriangleup B_j = A_j \backslash B_j = A_j \backslash B = A_j \cap M$$

is also included in the finite union of rows and columns. This proves that the ideal $\mathcal{I}_2$ has the property $(AP2)$.

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